# Wind Turbine Aerodynamics: Theory of Drag and Power

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This paper explores the mathematical models of the aerodynamics of wind turbines, focusing on wind drag and power production. The first theory, Actuator Disk Theory, provides a metric for studying wind turbine performance as well as an upper-limit for power production, known as the Betz Limit. The second theory, Blade Element Theory, utilizes airfoil theory to describe the lift and drag on the turbine blades. Together, these two models describe the Blade Element Momentum Theory, a powerful computational tool for the designing and testing of wind turbines.

Wind turbines have been in use since the tenth century [1], however the mathematical models describing their energy conversion were only formulated in the past century. In an effort to improve wind turbine efficiency, present research builds off of the original mathematical formulations of the nineteenth century. Current research includes improving theoretical concepts such as limits of power efficiency [2–5], computational models [6–9], and environmental effects of wind turbine farms [10–12]. The goal of this paper is to introduce the models that motivate the current research in wind energy and turbine design, as well describe the Blade Element Momentum Theory, a powerful tool for designing wind turbines.

The first model for understanding wind turbine aerodynamics and power output were formulated by Rankine and Froude [13–15] in their studies of propeller thrust dynamics. While Rankine and Froude's Actuator Disk Theory (also known as the axial momentum theory) was initially formulated for propellers, the observation that wind turbine dynamics share similar characteristics lends to a similar analysis for turbines. We notice that propellers outputs a fluid thrust (air or water) with an input of power through some fuel source (electricity or gasoline). Turbines function in the exact reverse: propellers output power in the form of electricity, and a thrust in the form of air drag acts against the turbine. With this simple comparison, we see that there is strong motivation to use the methods and theory of Rankine and Froude to initially analyze wind turbines.

The first simplification in the the Actuator Disk Theory is to extract out the details of the physical wind turbine and replace it with an actuator disk which serves as a semi-permeable disk which converts the wind energy into mechanical energy. In addition, we make the following initial assumptions: (1) the disk is frictionless; (2) there is no rotational velocity in the wake; (3) flow is stationary, incompressible, and frictionless; and (4) there are no external forces acting on the fluid up or downstream of the turbine. While these assumptions oversimplify certain regimes and dynamics, they provide us with an ideal model for the turbine, and we will revisit these assumptions with later corrections and modifications to this model.

The main strategy in modeling the turbine in the Actuator Disk Theory is to focus on the pressure and velocity drops that occur at the turbine. In order to do so, we introduce three different regions in the airflow: upstream with a velocity $u_\infty$ and pressure $p_\infty$, disk or turbine with velocity and pressure $u_d$ and $p_d$, and downstream, behind the turbine in its wake, $u_w$ and $p_w$. As shown in Figure 1, there is a sharp pressure drop across the wind turbine due to the decrease in wake area as well as a decrease in wind speed due to the loss of kinetic energy that is transferred to harnessed electrical energy.

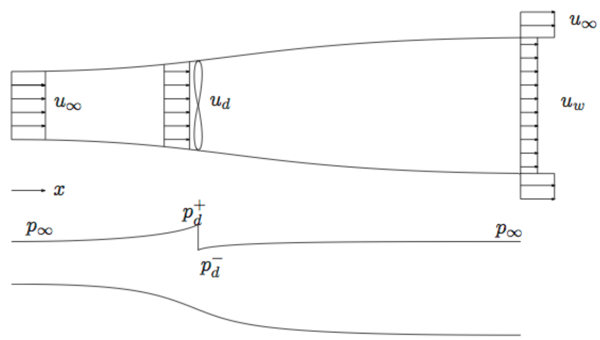


FIG. 1. Wind turbine modeled as actuator disk in Axial Momentum Theory. (Top) The flow profile for the upstream and downstream flow. The area of the streamlines are increased behind the disk. (Middle) Pressure distribution of the fluid, with a sharp decrease across the wind turbine. (Bottom) Velocity distribution for the fluid, varying from $u_\infty$ upstream to $u_d$ at the turbine/disk and finally $u_w$ in the downstream wake. [16]

We begin our analysis by formulating an expression for drag in terms of pressure, $T_p = \Delta p A$ where $\Delta p = p_d^+ - p_d^-$ is the drop in pressure over the disk and $A_d$ is the area of the disk or the swept area in the case of a wind turbine. In this paper, $T$ is chosen to represent the drag force to stay consistent with propeller notation, where the pressure drop over the propeller causes a thrust force in the opposite direction of the flow. The first challenge is to calculate this unknown pressure drop across the disk. In order to calculate the pressure drop we turn to Bernoulli's equation, describing the pressure distribution of the fluid between two points:

$$p + \frac{1}{2}\rho u^2 = \text{constant} \tag{1}$$

where scalar notation is used instead of vector notation since we assume the flow is one-directional, in the direction of the turbine axis.

Applying (1) to both the upstream and downstream portion of the wind flow and equating both sides we obtain an expression for the drop in pressure in terms of $u_\infty$ and $u_w$, $\Delta p = \frac{1}{2}\rho(u_\infty^2 - u_w^2)$. Substituting this equation into our expression for pressure drag, we obtain our first expression for the drag force on the disk: $T_p = \frac{1}{2}\rho(u_\infty^2 - u_w^2)A_d$. The second formulation is derived from the axial momentum theorem, $T_m = \dot{m}\Delta u$, where $\dot{m}$ is the mass rate. Applying the axial momentum theorem to the disk, we can obtain a thrust in terms of air speed: $T_m = \rho A_d u_d(u_\infty - u_w)$.

Equating the two expressions for drag, $T_p$ and $T_m$, we arrive at the interesting result that the air speed at the disk is the average of the wind speed and wake speed:

$$u_d = \frac{1}{2}(u_\infty + u_w) \tag{2}$$

Using the first law of thermodynamics, one can show that the total power extracted is [17]:

$$P = \frac{1}{2}\dot{m}(u_w^2 - u_\infty^2) = \frac{1}{2}\rho A_d u_d(u_w^2 - u_\infty^2) \tag{3}$$

In order to characterize the performance of a given wind turbine, we define performance constants calculated as ratios between actual force and total possible drag, and likewise, the ratio between the actual power output and total possible power output. Here we define the total possible force and power output as the force and power at the inlet, $u_\infty$. We choose the inlet since we know velocity is maximum there, and so too force and drag will be maximized. Thus, we define our drag and power performance coefficients as $C_d = \frac{D}{\frac{1}{2}\rho u_\infty^2 A_d}$ and $C_p = \frac{P}{\frac{1}{2}\rho u_\infty^3 A_d}$ respectively.

Before proceeding, it is helpful to introduce a term to quantify the reduction in air speed due to the interaction with the turbine/disk. The induction factor (or retardation factor), $a$, computes the ratio between the speed at the disk and the speed at the inlet, $u_\infty$ as [18]: $a = 1 - \frac{u_d}{u_\infty}$. Notice that for $a = 1$, the air is at rest at the turbine, and for $a = 0$ the air speed is not effected by the turbine. Using these factors, we seek to derive the theoretical limit for power performance by expressing our performance coefficients in terms of $a$, obtaining: $C_p = 4a(1-a)^2$ and $C_d = 4a(1-a)$. By taking the first derivative of the power coefficient, $C_p$ with respect to $a$, we can determine the maximum power possible by any turbine, $\frac{dC_p}{da}|_{a=a_{\max}} = 4(1-a_{\max})(1-3a_{\max}) = 0$.

As shown in Figure 2, there will be a maximum $C_p$ corresponding to an induction factor $a_{\max}$. Solving for $a_{\max}$ we obtain the optimization result that the optimal power output occurs at $a_{\max} = \frac{1}{3}$. At this axial induction factor, the maximum power coefficient is

$$C_{p,\max} = 4a_{\max}(1-a_{\max})^2 = \frac{16}{27} = 0.593 \tag{4}$$

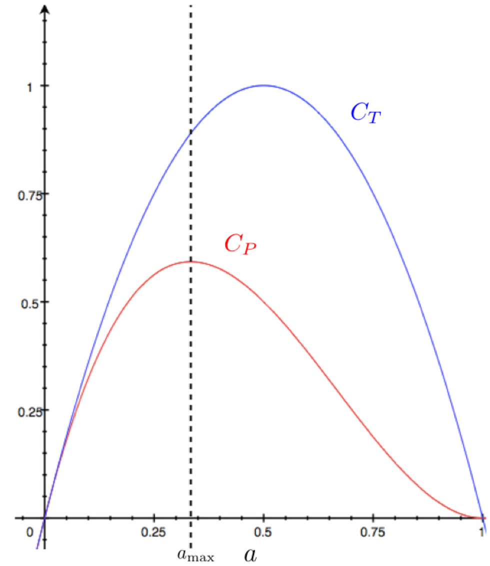


FIG. 2. Performance coefficients and Betz limit. $C_p$ (red) and $C_d$ (blue) graphed as a function of axial induction factor, $a$. The local maximum of $C_p$ is marked, known as the Betz limit ($a_{\max}$), for power performance of wind turbines.

This result, initially presented by Betz (1926) [19], provides an upper limit to the possible power coefficient for any wind turbine design. Following Betz, other researchers have refined this limit to match turbine data [3] and modern turbines today can achieve performance coefficients of approximately 0.5 [20].

Before concluding our discussion of the Actuator Disk Theory, we revisit the assumption that the flow is irrotational. While the air flow is irrotational in the upstream region, the interaction of the rotating blades will cause the air to rotate once it passes the turbine, creating a rotating wake. To quantify this rotation, we introduce a rotational induction factor, $a'$ that characterizes the rotational speed of the air relative to the rotational speed of the turbine blades: $a' = \frac{\omega}{2\Omega}$ where $\omega$ is the wake's rotational velocity and $\Omega$ is the turbine's rotational velocity. Like with the previous axial induction factor, the rotational induction factor describes the increase in rotational velocity due to the turbine. With this new contribution to both momentum and kinetic energy, we recalculate the forces and power due to the rotation of the turbine. We first recall Euler's Turbine equation which calculates power as a function of torque and rotational velocity instead of force and translational velocity $P = M_z\Omega = \Omega\dot{m}(r_1 C_{\theta,1} - r_2 C_{\theta,2})$ [17], $\dot{m}$ is the mass rate of the air, $r$ is the radius of the streamtube, and $C_\theta$ is the azimuthal velocity of the flow. Applying this to to an infinitesimal control volume of thickness $dr$ we obtain: $dP = \dot{m}\Omega r C_\theta = 2\pi r^2 \rho u_d \Omega C_\theta dr$ [17]. Since the rotational velocity in the wake ($rC_\theta$) contains rotational energy that was not extracted by the turbine, we want to minimize $C_\theta$. Thus for a given power output, maximizing the angular velocity of the wind turbine will minimize the energy loss due to wake rotation. Rewriting the power expression in terms of $a$ and $a'$ and integrating we obtain the new expression: $P = 4\pi\rho\Omega^2 u_\infty^2 \int_0^R a'(1-a)r^3 dr$. The power coefficient equation can then be written as $C_p = \frac{8}{\lambda^2}\int_0^\lambda (1-a)a'x^3 dx$ where $\lambda = \frac{\Omega R}{u_\infty}$ is the tip speed ratio and $x = \frac{\Omega r}{u_\infty}$ is the ratio between local rotational speed and the wind speed.

While the Actuator Disk Theory provides a useful modeling of the global conversion of wind speed and pressure gradients into power, including useful theoretical limits such as the Betz limit, the Actuator Disk Theory does not quite capture the entire mechanisms of wind turbine aerodynamics. Firstly, Actuator Disk Theory treats the wind turbine as a "Black Box" in its analysis. The physical description of the blade is not included in this model, nor are the physics of the blade included in the derivation. Secondly, this analysis does not comment on how to determine the axial induction factors without observing the flow velocity at the blade ($u_d$ and $\omega$), quantities that are not easily determined. Finally, and perhaps most importantly, Glauert showed empirically that this model fails to describe turbines with larger turbulent behaviors and axial induction factors. As shown in Figure 3, Glauert found that after $a = 0.5$, the thrust coefficient diverges from the Actuator Disk Theory.

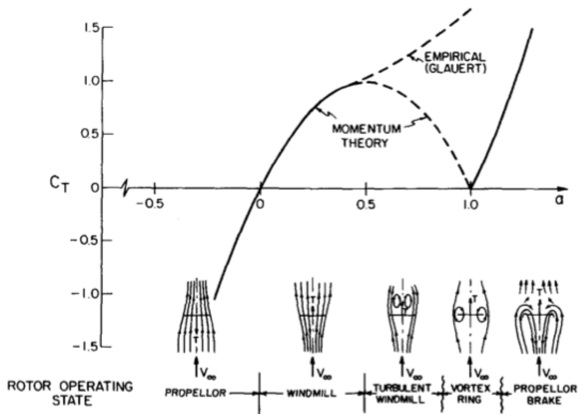


FIG. 3. Experimental data for drag coefficient as a function of axial induction factor. For dynamics with lower induction factors (propellers, windmills) Momentum/Actuator Disk Theory matches empirical results. For more turbulent conditions ($a > 0.5$), empirical data no longer matches the predicted theoretical values [21].

With the need for an alternative (and hopefully complementary) model of turbine dynamics, we focus on the blade itself in Blade Element Theory. Classic airfoil theory states that when a stream of fluid flows past an airfoil, two forces arise due to the pressure field created: a lift force and drag force. The lift force is perpendicular to the direction of the flow and drag force is parallel the flow. These forces arise from the varying pressure field around the airfoil as the fluid is deflected around the wing [17]. These forces can be shown to be functions of the geometry of the foil and the flow of the fluid:

$$L = \frac{1}{2} c_l \rho u^2 c \tag{5}$$

$$D = \frac{1}{2} c_d \rho u^2 c \tag{6}$$

where $c_l$ and $c_d$ are foil specific parameters called the lift and drag coefficients, $u$ is the fluid's velocity, $\rho$ is the density of fluid, and $c$ is the chord length (the chord line connects opposite ends of the wing) as seen in Figure 4b.

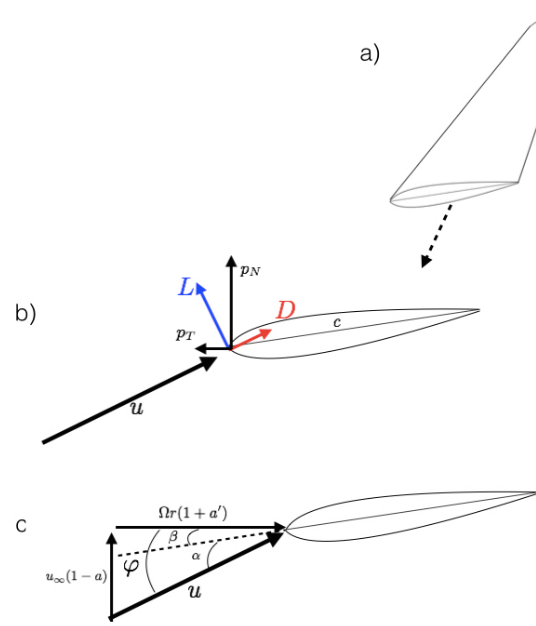


FIG. 4. Blade element diagram showing forces and flow angles. (a) Cross section of blade shows classic airfoil profile (b) Lift force perpendicular to flow and drag force parallel to flow. Decomposed into components normal, $p_N$ and tangential, $p_T$, to the rotor plane. (c) Flow angles due to turbine pitch and azimuthal and axial flow contributions. Relative wind angle $\varphi$ shown to the sum of $\alpha$ and $\beta$.

In the case of turbine blades, air is not flowing directly parallel to the airfoil's chord line and thus we introduce a parameter $\alpha$, or the angle of attack, of the blade. The angle of attack is important because the performance coefficients depend on the angle of attack and the Reynolds Number $= \frac{uc}{\gamma}$. For small angles of attack, the lift coefficient increases proportionally to the angle of attack, at a slope of roughly $2\pi$ (Figure 5a) and the drag coefficient is roughly constant (Figure 5b). At a certain angle, $C_l$ begins to drop dramatically in what is know as stalling [17], introducing higher drag effects and lower power production. Due to its influence of drag coefficients and stalling, angle of attack is the main parameter of interest in Blade Element Theory and thus an expression for this angle is of great importance.

In wind turbines, this angle of attack has two main contributions, pitch angle $\beta$ and relative wind angle $\varphi$. The

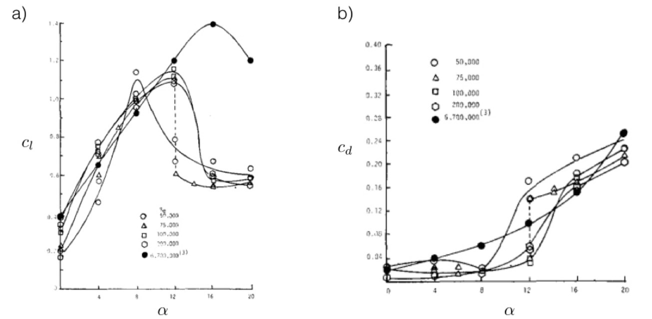


FIG. 5. Experimental data for lift and drag coefficients at various Reynold's Numbers. For small angles, lift coefficient varies linearly with angle and drag is constant. At the stall angle (dotted line), a sharp increase in drag and decrease in lift is observed. [22]

pitch angle measures the angle between the chord line and rotor plane and the relative wind angle measures the angle between the relative wind velocity ($u$). From Figure 4b, we see that there is a simple relationship between these three angles: $\alpha = \varphi - \beta$. The relative wind angle, $\varphi$, appears in our figure due to air flow coming from two directions relative to the turbine: an axial term that is purely from the wind ($u_d = u_\infty(1-a)$) and an azimuthal term due to the rotation of the turbine blades ($C_\theta = \Omega(1+a')$). From the geometry of these two air velocity contributions as seen in Figure 4b, an expression for the relative wind angle can be found as:

$$\tan\varphi = \frac{u_d}{C_\theta} = \frac{(1-a)u_\infty}{(1+a')\Omega r} \tag{7}$$

where $u_d$ is the axial wind velocity at the turbine, $C_\theta$ is the azimuthal component of the wake velocity, $u_\infty$ is the upwind air speed, and $\Omega$ is the angular velocity of the turbine. Note that $\varphi$ is a function of $r$, thus as one moves farther down the wing, $\varphi$ decreases. With this equation, we can calculate the angle of attack $\alpha$ purely from the wind speed and rotation of the blade, as well as the induction factors.

While we have obtained the blade's lift and drag forces, only components of these forces will be in the tangential direction (causing the drag thrust) and the normal direction (producing the torque for power output). However, using the geometry of the flow and wing, we can decompose the lift and drag forces into components normal and tangential to the rotor plane:

$$p_T = L\cos\varphi + D\sin\varphi \tag{8}$$

$$p_N = L\sin\varphi - D\cos\varphi \tag{9}$$

where $L$ and $D$ are the lift and drag forces calculated using equations (5) and (6). Since $p_T$ and $p_N$ are forces per length of the turbine, we integrate (8) and (9) along the blade to obtain the total drag force and torque, $T = \int Bp_N dr$ and $M = \int rBp_T dr$ where $B$ is the number of blades on the turbine.

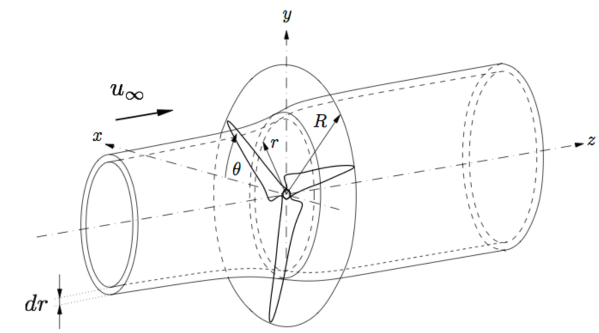


FIG. 6. Annulus of air with thickness $dr$ analyzed in Blade Element Momentum Theory. Pressure and velocity loss through annulus can be analyzed using Actuator Disk Theory. Lift and drag calculated from the flow over blade section using Blade Element Theory. [23]

In both Actuator Disk Theory and Blade Element Theory, our ability to calculate drag and power is dependent on our ability to measure the axial induction factors, $a$ and $a'$. Thus the next step is to look for a computational method to determine $a$ and $a'$, allowing us to fully describe the dynamics of any turbine design. Blade Element Momentum Theory provides a solution by combining both aerodynamic theories to formulate expressions for the induction factors. In this model, we analyze an annulus of air as it flows past a section of the blade as shown in Figure 6, with the assumption that the forces and torques on each individual annulus is radially independent and that the forces are constant along the annulus (corresponding to a turbine with infinite blades). By equating the results from both previous theories, we arrive at an expression for the induction factors:

$$a = \frac{1}{\frac{4\sin^2\alpha}{\sigma C_n} + 1} \tag{10}$$

$$a' = \frac{1}{\frac{4\sin\alpha\cos\alpha}{\sigma C_t} - 1} \tag{11}$$

where $\sigma$ is the blade solidity or fraction of annular area covered by the blades, and $C_n$ and $C_t$ are the dimensionless forms of $p_N$ and $p_T$. With these factors, we can now calculate the total drag and power generated by summing each blade element's contribution. Since $a$ and $a'$ can be found iteratively using (7), (10), and (11), this method is computationally inexpensive and thus used by many engineers to design and test wind turbines.

In summary, we have presented two models for calculating the power production and drag force. Actuator Disk Theory motivates the introduction of induction factors to describe the performance of ideal wind turbines and deduce theoretical limits to the power production.

The Blade Element Theory provides a description of the forces in terms of blade geometry and airfoil coefficients. Finally, Blade Element Theory combines both the blade specific dynamics of Blade Element Theory and the flow velocities of Actuator Disk Theory to provide a useful method for calculating the drag and power of a specific turbine design. Future work will concentrate on computational and experimental methods such as computational fluid dynamics and wind tunnel experimentation, which can be used to verify our theoretical models and provvie further insight on the dynamics of wind turbines.

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